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FRACTIONAL FREE ELECTRON LASER EQUATION AND GENERALIZED M-SERIES

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ABSTRACT

In this decade fractional free electron laser (FEL) equation are studied due to their utility and importance in mathematical physics, The aim of present work is to find the solution of generalized fractional order free electron laser (FEL) equation, using Generalized M-SERIES. The results obtained here is moderately universal in nature. Special cases, relating to the exponential function is also considered.

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RIEMANN-Liouville operator.

INTRODUCTION

The Fractional Free Electron Laser Equation:

The unsaturated behaviour of the free electron laser (FEL) is governed by the following first order integro differential equation of Volterra – type [3,4]. :

$$D_T\ a(T) = -i\pi\ g_0 \int_0^T \xi\ a\ (T - \xi)\ e^{iv\xi}\ d\xi, \quad 0 \le T < 1 \qquad \text{...(1.1)}$$

where T is a dimensionless time variable, g_0 is a positive constant known as the small–signal gain and the constant v is the detuning parameter. The functional (T) is a complex-field amplitude which is assumed to be dimensionless and satisfies the initial condition a (0) = 1. Here we employ the Riemann-Liouville definition of fractional integral equation defined by a simplified version of (1.1) changing the scale by putting $t = x\sigma$ and a = 0 this yields

$$R_x^\alpha\ f\ (x) \equiv I_x^\alpha\ f\ (x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\ f\ (x\,\sigma)\ d\sigma, \quad \mathrm{Re}\,\sigma \ge 0$$

... (1.2)

The definition (1.2) can be written as

$$R_x^\alpha\ f\ (x) \equiv I_x^\alpha\ f\ (x) = \frac{d^n}{dx^n} R_x^{\alpha+n}\ f\ (x), \qquad \mathrm{Re}\,(\alpha + n) > 0 \qquad \text{...(1.3)}$$

Boyadjiev et al. [3] have treated a non homogeneous case of (1.2) in which the ordinary first derivative D_T is replaced by the fractional D_T^α with $\alpha > 0$, that is

$$D_T^\alpha\ a\ (T) = \lambda \int_0^T t\ a\ (T - t)\ e^{ivt}\ dt + \beta\ e^{ivt}, \qquad 0 \le T \le 1 \qquad \text{...(1.4)}$$

with $\beta, \lambda, \in \mathrm{C}$ and $\mathrm{v} \in \mathrm{R}$. Furthermore the following generalization of (1.4) has been considered by the authors [2]

$$D_T^{\alpha}\, a(\mathrm{T}) = \lambda \int_0^{\mathrm{T}} t^{\delta}\, a(\mathrm{T}-t)\, e^{ivt}\, dt + \beta\, e^{ivt}, \quad 0 \le \mathrm{T} \le 1 \qquad \ldots(1.5)$$

where $\beta, \lambda, \in \mathrm{C}$, $\mathrm{v} \in \mathrm{R}$ and $\delta > -1$, In the present section, we investigate a further generalization of equation (1.5), whereby the exponential term is replaced by the M-SERIES **2. The Generalized Equation:**

The generalization of equation (1.5) obtained by replacing e^{ivt} by $_p\overset{\alpha,\beta}{\mathrm{M}}_q(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + \beta)}$ takes the form

$$D_T^{\alpha}\, a(\mathrm{T}) = \lambda \int_0^{T} t^{\delta}\, a(\mathrm{T}-t)\, {}_p\overset{\alpha,\beta}{\mathrm{M}}_q(ivt)\, dt + \beta\, {}_p\overset{\alpha,\beta}{\mathrm{M}}_q(ivt), \qquad \ldots(2.1)$$

$$0 \le \mathrm{T} \le 1.$$

where $\alpha, \beta, \lambda \in \mathrm{C}; \mathrm{v} \in \mathrm{R}$, $\alpha > 0$, $\delta > -1$, Given real numbers b_k, the corresponding initial conditions are :

$$D_T^{\alpha-k}\, a(\mathrm{T}) \Big|_{\mathrm{T}=0} = b_k, \quad k = 1, 2, 3, \ldots, N \qquad \ldots(.2.2)$$

with $N = [\alpha] + 1$, so that $N - 1 \le \alpha < N$. Equation (1.5) can be deduced from (2.1) by taking $\alpha = 1$ while (2.4) follows when $\alpha = 1$, $\delta = 1$ To obtain the solution of (2.1) for the given initial conditions (2.2), we use (2.2).

Let $\xi = \mathrm{T} - t$ in (2.1), so that $D_T^{\alpha}\, a(\mathrm{T}) = \lambda \int_0^{T} (\mathrm{T}-\xi)^{\delta} a(\xi)\, {}_p\overset{\alpha,\beta}{\mathrm{M}}_q (iv(t-\xi)d\xi + \beta_p\, \overset{\alpha,\beta}{\mathrm{M}}_q\, (iv\mathrm{T})$

…(2.3)

Using the series representation for $_p\overset{\alpha,\beta}{\mathrm{M}}_q$ we get

$$a(T) = a_0(\mathrm{T}) + \lambda\, I_{\mathrm{T}}^{\alpha} \left[\sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{(iv)^k}{\Gamma(\alpha k + \beta)} \int_0^{\mathrm{T}} (\mathrm{T}-\xi)^{\delta+k}\, a(\xi)\, d\xi \right]$$
$$+ \beta\, I_{\mathrm{T}}^{\alpha} \left[\sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{(iv)^k}{\Gamma(\alpha k + \beta)} \right]$$

...(2.4)

where $a_0(\mathrm{T}) = \sum_{k=1}^{N} \frac{b_k}{\Gamma(\alpha - k + 1)} \mathrm{T}^{N-k}$ …(2.5)

$$\int_0^t\int_0^T u\,(\mathrm{T,s})\,\mathrm{ds\,dT} = \int_0^t\int_s^t \mathrm{u}\,(\mathrm{T,s})\,\mathrm{dT\,ds}$$

…(2.6)

We obtain the following result

$$a\,(\mathrm{T}) = \mathrm{a}_0\,(\mathrm{T}) + \frac{\lambda}{\Gamma(\alpha+1)}\int_0^T \mathrm{a}\,(\xi)\,(\mathrm{T}-\xi)^{\alpha+\delta}\;{}_p\overset{\alpha,\beta}{\mathrm{M}}_{\mathrm{q}}\;(\mathrm{i\,v\,(T-}\xi))\mathrm{d}\xi$$

$$+\frac{\beta\,\Gamma(\mathrm{k}+1)\,\mathrm{T}^{\alpha}}{\Gamma(\alpha+k+1)}\,{}_p\overset{\alpha,\beta}{\mathrm{M}}_{\mathrm{q}}\quad(\mathrm{ivT})$$

...(2.7)

Since (2.7) is a Volterra integral equation with continuous kernel, it admits a unique continuous solution (2.4). Finally, we consider some special cases of the generalized fractional integro-differential equation of volterra – type (2.1)

1. If $\alpha = 1, \beta = 1$ and there is no upper and lower parameter

$$D_T^{\alpha}\;\mathrm{a\,(T)} = \lambda\int_0^{\mathrm{T}} t^{\delta}\;\mathrm{a\,(\,T-t)\,e}^{\mathrm{iv\,(T-}\xi)\,\mathrm{d}\xi} + \frac{\beta\,\Gamma(\mathrm{k}+1)\,\mathrm{T}^{\alpha}\;\mathrm{e}^{\mathrm{ivt}}}{\Gamma(\alpha+\mathrm{k}+1)}$$

...(2.8)

Equivalently

$$D_T^{\alpha}\;\mathrm{a(T)} = \lambda\int_0^{\mathrm{T}} t^{\delta}\;\mathrm{a\,(T-t)}\;{}_0\mathrm{F}_0\;(\text{-;-;}\mathrm{iv\,(T-}\xi)) +$$

$$\beta\,\frac{\Gamma(\mathrm{k}+1)}{\Gamma(\alpha\;+\mathrm{K}+1)}\,\mathrm{T}^{\alpha}\;{}_0\mathrm{F}_0\;(\text{-;-;}\mathrm{ivT})\qquad ...(2.9)$$

CONCLUSION

In this present work, we have introduced a fractional generalization of the standard free electron laser (FEL) equation The results of the Advanced generalized fractional free electron laser (FEL) equation and its special cease are same as the results of Al-Shammery, A, Kalla, and Khajah, [2](2003).

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